APPENDIX : INTEGRATION IN PRACTICE

Closed form evaluation of the integral $z = \int_A^B f(x)\,dx$ is feasible only for a very limited class of simple applications; in practice, numerical or graphical techniques must usually be employed.

You might prepare a numerical integration procedure for your calculator or computer. Such a procedure could be based on recursive subdivision using Simpson's method. To increase robustness it might embody the transformation $x = m\,y\,(3 - y^2) + n$ where m, n are constants chosen so that the newly introduced independent variable, y, assumes a minimum value of -1 when x = A; and a maximum value of +1 when x = B; ie $m = (B-A)/4$; $n = (B+A)/2$. In terms of y, the integral then assumes the form :-

$$z = 3m \int_{-1}^{1} (1 - y^2) \text{ function } (\, m\,y\,(3 - y^2) + n \,)\, dy$$

The transformed integrand ***automatically*** vanishes at the two limits, so it does not need to be evaluated there and the transformation avoids potential difficulties due to the fairly common occurrence of a singularity at either limit. This transformation is incorporated into the Romberg integration procedure in the program *Motors*. Lacking such procedures, functions may be graphed and the following applied.

Graphical Integration

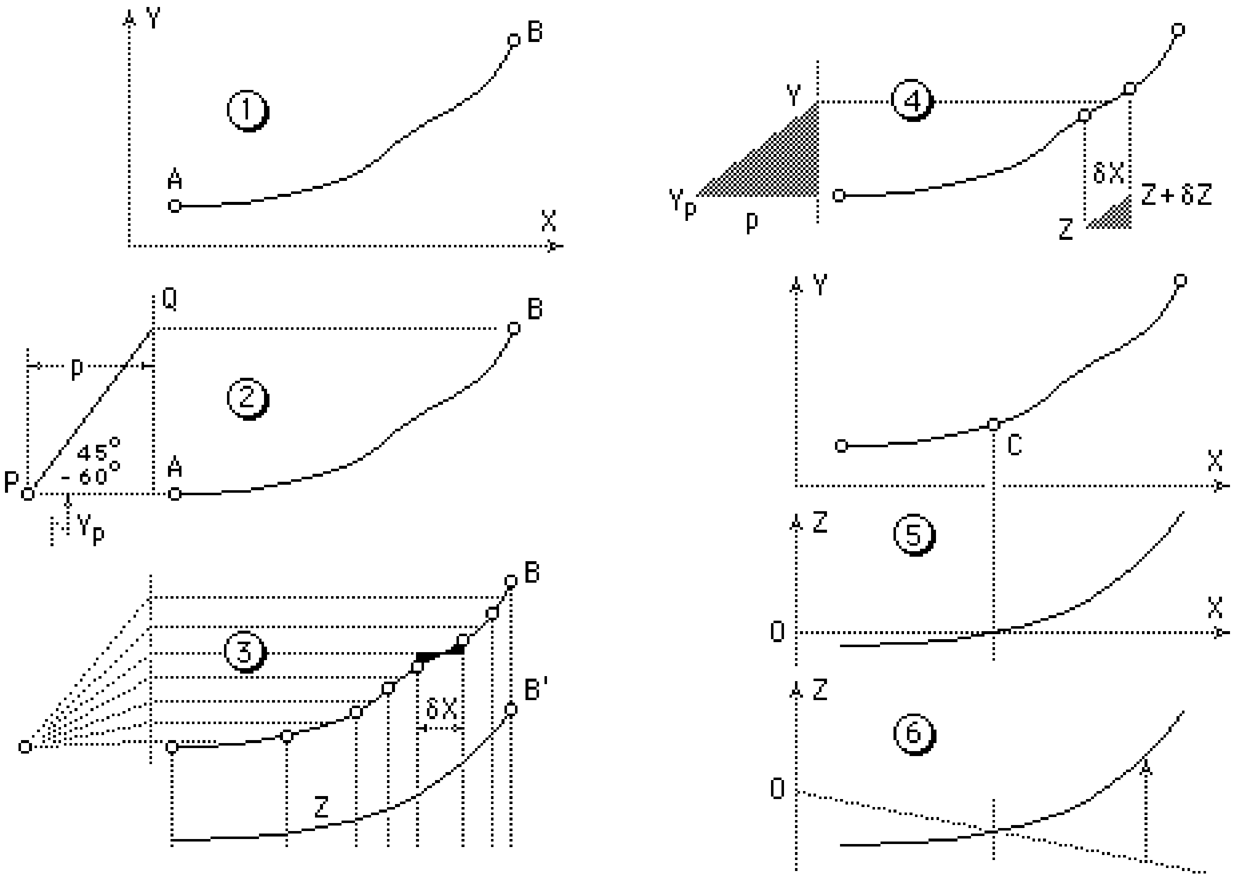
To evaluate graphically the integral

(i) $\quad z = \int_A^B y\,dx \quad$ where x and y may be temperature, or stress, or length, or whatever.

The y-x relation is plotted as a Y-X graph, both X and Y being ***lengths*** in Figure 1 for example, drawn to known scales :

(ii) $\quad S_x = x/X$ (eg - units of 'x' per mm) ; $\quad S_y = y/Y$ (units of 'y' per mm)

The following construction gives the integral, Z, also as a dimension, to some scale, S_z, as yet unknown.



Choose any line Q parallel to the Y-axis, and a pole P, with ordinate Y_p and distant p (mm) from Q. In Figure 2, P has been taken with the same ordinate as A, but this is not necessary.

Divide the curve into a number of vertical strips, Figure 3, and for each strip :

- select the mid-point
- construct a line parallel to the X-axis from the mid-point to the line Q and thence to the pole, P
- parallel to this polar line, draw a line across the strip from the similar line in the previous strip.

This construction would start from any point such as B' having the same abscissa as one of the limits, and yields the Z-curve, from which the integral follows. Thus from similar triangles in Figure 4 :

$$(Y - Y_p)/p = ((Z + \delta Z) - Z)/\delta X \quad ; \qquad \delta Z = ((Y - Y_p)/p)\,\delta X \qquad \text{or, integrating}$$

$$Z = {}^1/_p\left(\int Y\,dX - Y_p X + \text{constant}\right) \qquad \text{- and using (ii)}$$

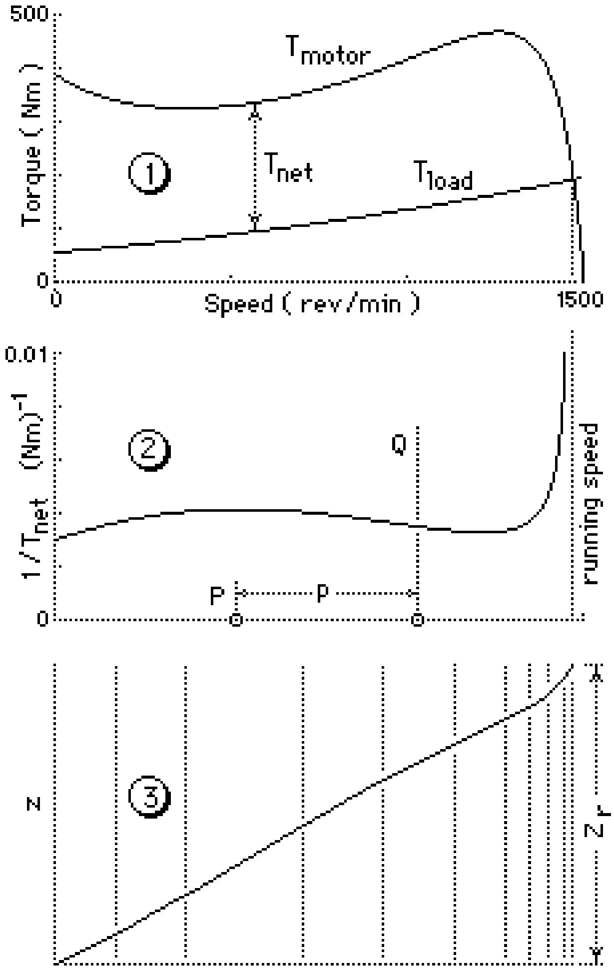
$$z/S_z = \left(\int y\,dx - y_p x - z_o\right)/p\,S_x S_y \qquad \text{- where } z_o \text{ is constant; then setting . . . .}$$

(iii) $S_z = p\,S_x\,S_y$ (mm $*$ (units of x per mm)$*$ (units of y per mm)) . . . . leads to

(iv) $\int y\,dx = z + z_o + y_p x$ - which, with limits, yields

$$\int_A^B y\,dx = z_B - z_A + y_p(x_B - x_A) \qquad \text{. . . . the desired result.}$$

If P is chosen at Y = 0, then the last term of (iv) vanishes and (i) is satisfied identically. If this be the case, and furthermore the integral is known to vanish at some point C, then the plot of the integral may be completed by the addition of the X-axis as shown in Figure 5. But this is not always immediately possible - sometimes a second integration is necessary to simultaneously establish two integration constants; the closed form integration of the simple beam's elastic curve is a case in point. If Y_p<> 0 then the integral must be reckoned from the inclined line inferred by (iv) and indicated in Figure 6.



EXAMPLE 1 A load is characterised by constant, linear and quadratic components of 55, 80 and 60 Nm respectively, at 1500 rev/min. The load's inertia is 25 kg.m^2 and it is driven directly by a now superseded ASEA squirrel cage motor, type MBN 200 L. Estimate the acceleration time graphically.

The motor and load torques are plotted against speed, Figure 1, using a speed scale of :

$$S_n = 25\,(\text{Hz}) / 47\,(\text{mm})$$

The net torque is computed at various speeds and its reciprocal plotted, Figure 2, with a scale of :

$$S_T = 0.01\,(\text{Nm})^{-1} / 23\,(\text{mm})$$

A pole, P, and reference line, Q, are conveniently selected on Figure 2, with pole distance p = 16 mm.

The integral is then drawn, Figure 3, whose value up to running speed is measured as Z_r = 26 mm.

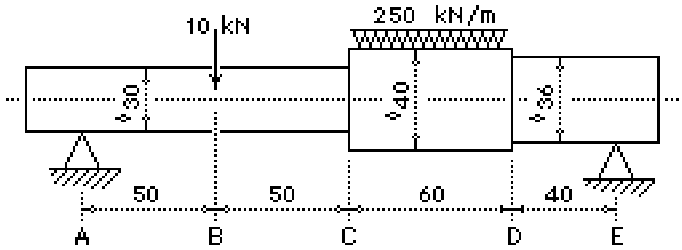
The acceleration time is therefore :

$$\begin{aligned}\Delta t &= 2\pi J \int_0^{n_r} (1/T_{net})\,dn = 2\pi J\ z_r \\ &= 2\pi J\,(S_z\ Z_r\ ) = 2\pi J\,(p\,S_n\ S_T\ )\,Z_r \\ &= 2\pi\,(25{+}0.35)\ 16\ (25 / 47)\ (0.01 / 23)\ 26 * 1 \\ &\quad \frac{\text{rad}}{\text{rev}}\ \ \text{kg m}^2\ \ \text{mm}\ \ \frac{\text{rev}}{\text{s mm}}\ \ \frac{1}{\text{Nm mm}}\ \ \text{mm}\ \ \frac{\text{N s}^2}{\text{kg m}} \\ &= \underline{15\ \text{s}}\end{aligned}$$

This value is comparable with the period calculated by the program *Motors*, though drawing inaccuracies become relatively large as the running speed is approached, and the program is based upon current motor characteristics.

Note that the Z = 0 axis is horizontal here as the pole is selected on the zero-ordinate axis.

EXAMPLE 2 Determine graphically the maximum deflection of the steel shaft illustrated.



Static analysis yields the bending moment diagram of Figure 1. Preparing the M/EI diagram :

$$EI_{ABC} = (\pi/64) * 0.030^4 * 207e9 = 8230\ \text{Nm}^2$$

$$EI_{CD} = (\pi/64) * 0.040^4 * 207e9 = 26012\ \text{Nm}^2$$

$$EI_{DE} = (\pi/64) * 0.036^4 * 207e9 = 17067\ \text{Nm}^2$$

Hence the M / EI diagram is plotted, Figure 2, to scales :

S_x = 200 (mm) / 48 (mm) = 4.17 mm / mm

$S_{M/EI}$ = 40 (mrad / m) / 15 (mm) = 2.67 mrad / m.mm

The pole P_θ is selected at zero ordinate and at distance p_θ = 11 mm as shown in Figure 2. Graphical integration then follows, yielding the slope curve (θ-x : Figure 3) to a scale, from (iii) of :

S_θ = 11 (mm)∗4.17∗2.67 (mrad / m.mm) = 122 mrad / mm

Although the zero axis of the slope curve is undetermined at this stage, it is known to be horizontal since the pole had been chosen at zero ordinate in Figure 2. Integration of the slope yields the deflection, v, so a pole P_v is chosen at a distance p_v = 9 mm in Figure 3, and the v-x curve constructed, Figure 4, whose scale, from (iii) is :

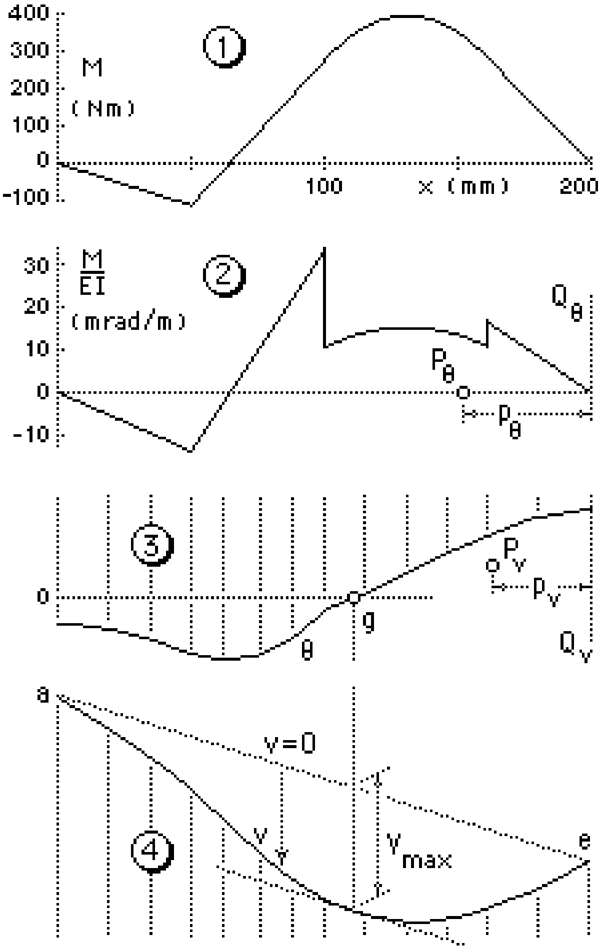
S_v = 9 (mm) ∗ 4.17 ∗ 122 (mrad / mm) = 4.59 mm / mm

Since the ordinate of the pole was not zero in Figure 3, the v = 0 axis will be inclined. It is known that there can be no deflection of the shaft supports, A & E, so the line a-e must represent the undeflected axis. A line parallel to a-e and tangent to the curve identifies the maximum deflection, which occurs at the point g. As this point is also the point of zero slope, the zero axis in Figure 3 can now be defined from the v-x curve.

From the deflection curve Figure 4 the scaled maximum deflection is measured as V_{max} = 11 mm.

The maximum deflection is therefore :

v_{max} = 11 (mm) ∗ 4.59 (mm / mm) = <u>50 mm</u>



In this example, two integrations are necessary before any integration constants can be evaluated. Analytic approaches to the bending of simply supported beams demonstrate this same necessity.

Accuracy will obviously be improved by graphs larger than the above, and by measurements better than to the nearest millimetre.