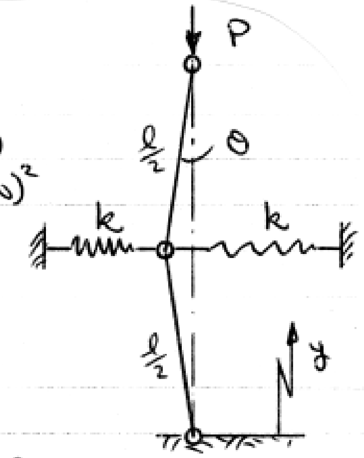

Height of load over datum = $l\cos\theta$
∴ P.E. gain = $Pl(\cos\theta - \cos\theta_0)$.
Deflexion of a spring = $\frac{l}{2}(\sin\theta - \sin\theta_0)$
∴ S.E. gain of both springs = $2.\frac{1}{2}k\frac{l^2}{4}(\sin\theta - \sin\theta_0)^2$
Total energy = $\frac{1}{4}kl^2(\sin\theta - \sin\theta_0)^2 + Pl(\cos\theta - \cos\theta_0)$
Set U = energy$/\frac{1}{2}kl^2$ & set $p = 2P/kl$

$$= \frac{1}{2}(\sin\theta - \sin\theta_0)^2 + p(\cos\theta - \cos\theta_0).$$

① $U' = (\sin\theta - \sin\theta_0)\cos\theta - p\sin\theta$

② $U'' = \cos 2\theta + \sin\theta_0 \sin\theta - p\cos\theta$

PERFECT $\theta_0 = 0$ – critical buckling at p_c.

$U' = \sin\theta(\cos\theta - p)$ from ①.
$U'' = \cos 2\theta - p\cos\theta$ from ②.

$U' = 0$ for equilibrium, so

③ either $\sin\theta = 0$, $\theta = 0$ – in which case $U'' = 1 - p$.
neutral $U'' = 0$, $p = 1$ – this is critical, i.e. $p_c = 1$
stable U'' +ve, $p < 1$ i.e. $p < p_c$
unstable U'' –ve, $p > 1$ i.e. $p > p_c$
critical load $P_c = \frac{1}{2}kl$.

④ or $p = \cos\theta$ – in which case $U'' = -\sin^2\theta$.
i.e. this is unstable (U'' –ve) for all θ.

There are therefore three postbuckling paths, all of which are unstable – $\theta = 0°$ and $\theta = \cos^{-1}p$ (θ is positive or negative).

IMPERFECT $|\theta| \geq |\theta_0| > 0$ – no critical but possible $\hat{p}$.

$U' = 0$ for equilibrium, so, from ①:

⑤ $p = (1 - \sin\theta_0/\sin\theta)\cos\theta$

Note that, mathematically, $p > 0$ for all $|\theta| > |\theta_0|$.

Inserting ⑤ into ②:

⑥ $U'' = (\sin\theta_0 - \sin^3\theta)/\sin\theta$ for the equilibrium trajectory.

Maximum p – i.e. $\hat{p}$ – will occur when $p' = 0$, i.e. when $U'' = 0$ – that is when equilibrium is neutral. In this case, from ⑥ $\sin^3\theta = \sin\theta_0$, so that, from ⑤:

⑦ $\hat{p} = [1 - (\sin\theta_0)^{2/3}]^{3/2}$.

– which is superimposed on the trajectories overleaf.

Considering positive θ, θ_0, from ⑥:
$\sin^3\theta < \sin\theta_0$, $U'' > 0$, stable.
$\sin^3\theta > \sin\theta_0$, $U'' < 0$, unstable.
which are exemplified in the trajectories' slopes.

Note that the trajectories are shown also for negative p corresponding to a tensile force necessary for $|\theta| < |\theta_0|$

contd...