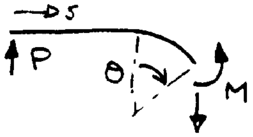
1 Apply Castigliano's to half the spring



straight portion $M = Ps \quad \partial M/\partial P = s$

curved portion $M = P(L + R\sin\theta)$

$$\partial M/\partial P = L + R\sin\theta$$

Since EI is constant

$$EI\,\delta_p = \int_0^L Ps.\,s\,ds + \int_0^{\frac{\pi}{2}} P(L+R\sin\theta)^2.\,R\,d\theta$$

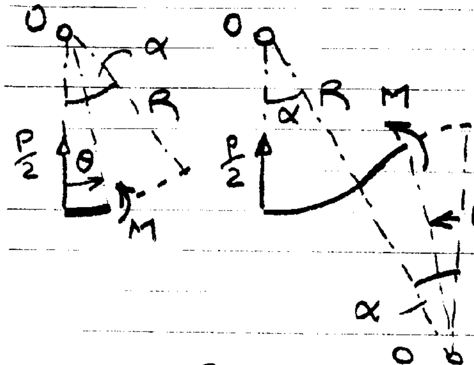
$$EI\,\delta_p/P = \tfrac{1}{3}L^3 + \left(\tfrac{\pi}{2}L^2R + 2LR^2 + \tfrac{\pi}{4}R^3\right)$$

$$= \tfrac{1}{12}\left(4L^3 + 6\pi L^2R + 24LR^2 + 3\pi R^3\right)$$

As the deflexion for the whole spring will be twice that of cantilever (i.e. $2\,\delta_p$) and as $I = \frac{\pi}{64}d^4$

$$\text{compliance} = 2\delta P/P = (4L^3 + 6\pi L^2R + 24LR^2 + 3\pi R^3)\,32/3\pi E d^4$$

2 Approach – use Castigliano's, recognising symmetry.



For the section to the left of the line of centres O-O

$$M = \tfrac{1}{2}P.\,R\sin\theta$$

& for the free body to the right of O-O:

$$M = \tfrac{1}{2}PR\,(2\sin\alpha - \sin\theta).$$

By symmetry about the load line, the strain energy of the system is:-

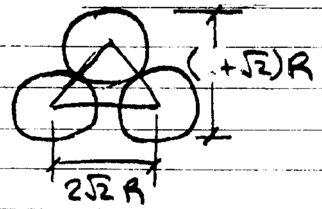
$$U = 2\,\frac{R}{2EI}\left[\int_0^\alpha \left(\tfrac{1}{2}PR\sin\theta\right)^2 d\theta + \int_0^\alpha \left(\tfrac{1}{2}PR(2\sin\alpha - \sin\theta)\right)^2 d\theta\right]$$

$$= \frac{P^2R^3}{2EI}\int_0^\alpha \left[2\sin^2\alpha - 2\sin\alpha\sin\theta + \sin^2\theta\right] d\theta$$

$$\delta_p = \frac{\partial U}{\partial P} = \frac{PR^3}{4EI}\left[3(2\alpha + \sin 2\alpha) - 4\alpha\cos 2\alpha - 8\sin\alpha\right].$$

QED.

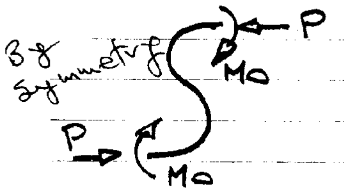
3



Consider superposition:

– firstly, unrestrained thermal expansion as of $L\alpha T$, eg. for symmetric half:

$$\delta = \tfrac{1}{2}(2\sqrt{2}R).\,\alpha T = \sqrt{2}\,R\alpha T.$$

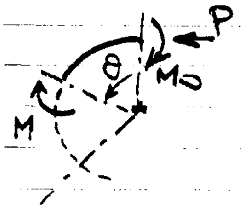


– secondly, imposition of P & Mo as shown on LH half to restore deflections to zero.

For equilibrium of LH half:

$$2M_o - P \times (2+\sqrt{2})R = 0$$

i.e. $M_o = (1 + 1/\sqrt{2})PR$.



Recognising symmetry of the LH 'S', we may integrate energy only over the top half as shown:

$$M = PR(1 - \cos\theta) - M_o \quad \text{\& from above}$$

$$= -PR(\cos\theta + 1/\sqrt{2})$$